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LETTER TO THE EDITOR

Langevin equation with back-reaction

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Abstract
We investigate a simple model of an overdamped Brownian particle in a harmonic potential. The dynamics is described by a Langevin equation with a two-valued stochastic force. The properties of the Langevin force are controlled through a back-reaction mechanism by the particle dynamics. Our analysis reveals two quite different dynamical regimes. The low-temperature regime exhibits a dynamically induced ergodicity breaking. However, an arbitrarily small periodic perturbation is capable of bringing the system back to ergodicity.

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The glass transition is an old problem which still abounds with unresolved questions. It is now widely believed that the phenomenon is inherently dynamical, i.e. it is the full solution of the dynamical equations that should be related to the experimental data. Several approaches have already proved useful, the mode-coupling theory [1] being perhaps the most popular one. In this letter, we introduce a very simple model based on the Langevin equation for a particle under the influence of a stochastic reservoir. The properties of the stochastic force are assumed to depend on the motion of the particle itself. We call this mechanism *back-reaction*.

We start with the Langevin equation for an overdamped Brownian test particle which moves in a harmonic potential

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{X}(t) = -\gamma\mathsf{X}(t) + \mathsf{Q}(t). \tag{1}$$

The stochastic force $\mathsf{Q}(t)$ is usually taken as Gaussian white noise [2,3]. Here, however, as in [4–6], it is assumed to be dichotomic noise [7]. It jumps between just two values $\pm q$. In [4–6], the jumps are generated with a *time-independent* rate $\frac{1}{2}\lambda_c$. As a result, the particle slides down a parabolic potential which switches at random instants between the shapes $\Phi_\pm(x) = \frac{1}{2}\gamma x^2 \mp qx$. The minima of the two parabolas are situated at $x_\pm = \pm q/\gamma$ and they define an attractive region $[x_-, x_+]$ which supports the asymptotic probability density for the particle position.

Already this simple problem is 'less trivial than it looks' [5]. The asymptotic probability density develops a noise-induced transition [7] depending on the ratio λ_c/γ. If this ratio is larger (smaller) than 2 [6], the function $\lim_{t\to\infty} p(x,t)$ is a convex (concave) function. Nevertheless, this constant-rate model still rests on the usual cornerstone of stochastic dynamics: the properties of the environmental force are fixed and they cannot be back-influenced by the test particle.

In order to incorporate into equation (1) a possible back-reaction, we use a stochastic force $\mathsf{Q}(t)$ which jumps between two values $\pm q$ at random instants generated by an underlying *time-nonhomogeneous* Poisson point process [8, 9]. The point process is defined by the time-dependent intensity $\frac{1}{2}\lambda(t)$ (the factor $\frac{1}{2}$ is included here for convenience). For the moment, let this function be given. In other words, the force $\mathsf{Q}(t)$ is a time-nonhomogeneous Markov process and the probabilities $\pi_\pm(t) = \mathrm{Prob}\{\mathsf{Q}(t) = \pm q\}$ are dictated by the Pauli master equation with *time-dependent* rates $\frac{1}{2}\lambda(t)$. Solving this equation, i.e. calculating the corresponding time-ordered exponential, one achieves a complete description of the noise. For example, the mean value $\kappa(t) = \langle \mathsf{Q}(t)\rangle$ and the correlation function $r(t,t') = \langle \mathsf{Q}(t)\mathsf{Q}(t')\rangle$ are controlled by the integral $\Lambda(t) = \int_0^t \mathrm{d}t'\lambda(t')$. More explicitly,

$$\kappa(t) = q\delta_0 \exp[-\Lambda(t)], \qquad r(t,t') = q^2 \exp[-|\Lambda(t) - \Lambda(t')|], \tag{2}$$

where $\delta_0 = \pi_+(t=0) - \pi_-(t=0)$ represents the initial condition for the noise. Notice that the mean force $\kappa(t)$ *either* relaxes to zero (e.g. if the function $\lambda(t)$ asymptotically approaches a positive constant) *or* to a non-zero constant (provided the function $\lambda(t)$ relaxes sufficiently rapidly to zero).

Now, the crucial new assumption of our model is that the rate function $\frac{1}{2}\lambda(t)$ depends on the particle motion. Intuitively, the immediate mean frequency of the environmental-force jumps should be smaller if the particle moves at a lower velocity. This reasoning leads to the back-reaction coupling of the form

$$\lambda(t) = \epsilon \frac{\gamma}{q^2} \langle \mathsf{V}^2(t)\rangle, \tag{3}$$

where ϵ is a dimensionless control parameter and $\mathsf{V}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathsf{X}(t)$ is the velocity. The bigger the mean kinetic energy of the particle, the bigger the instantaneous mean frequency of the force (i.e. the bigger the local 'temperature' of the environment).

In order to derive a closed set of equations, suppose again, for the moment, that the function $\lambda(t)$ is given. Performing a direct averaging in the Langevin equation (1), we express the mean position $\mu(t) = \langle \mathsf{X}(t)\rangle$ as

$$\mu(t) = x_0 \exp(-\gamma t) + q\delta_0 \int_0^t \mathrm{d}t' \exp[-\gamma(t-t') - \Lambda(t)]. \tag{4}$$

Here and below, we assume that all paths start at the same value $\mathsf{X}(t=0) = x_0$. Similarly, one can evaluate the second moment of the particle position and the first two moments of the particle velocity. These moments can be conveniently expressed as combinations of the following four auxiliary functions:

$$s_1(t) = \exp[-\Lambda(t)], \tag{5a}$$

$$s_2(t) = \exp(-\gamma t) \int_0^t \mathrm{d}t' \exp[\gamma t' - \Lambda(t')], \tag{5b}$$

$$s_3(t) = \exp[-\gamma t - \Lambda(t)] \int_0^t \mathrm{d}t' \exp[\gamma t' + \Lambda(t')], \tag{5c}$$

$$s_4(t) = \exp(-2\gamma t) \int_0^t \mathrm{d}t' \exp[\gamma t' - \Lambda(t')] \int_0^{t'} \mathrm{d}t'' \exp[\gamma t'' + \Lambda(t'')]. \tag{5d}$$

For example, the first moment of the position is $\mu(t) = x_0 \exp(-\gamma t) + q\delta_0 s_2(t)$ and the second one is $\langle \mathsf{X}^2(t)\rangle = x_0^2 \exp(-2\gamma t) + 2x_0 q\delta_0 \exp(-\gamma t) s_2(t) + 2q^2 s_4(t)$.

At this point, we employ equation (3) and we obtain the expression

$$\frac{1}{\epsilon}\frac{\lambda(t)}{\gamma} = 1 - 2\gamma[s_3(t) - \gamma s_4(t)] + \tilde{x}_0^2 \exp(-2\gamma t) - 2\tilde{x}_0\delta_0 \exp(-\gamma t)[s_1(t) - \gamma s_2(t)], \qquad (6)$$

where $\tilde{x}_0 = x_0\gamma/q$. The functions (5*a*)–(5*d*) obey the following system of differential equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}s_1(t) = -\lambda(t)s_1(t), \qquad (7a)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}s_2(t) = -\gamma s_2(t) + s_1(t), \qquad (7b)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}s_3(t) = 1 - [\gamma + \lambda(t)]s_3(t), \qquad (7c)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}s_4(t) = -2\gamma s_4(t) + s_3(t), \qquad (7d)$$

where the function $\lambda(t)$ must be substituted from equation (6). Thus the right-hand sides contain quadratic nonlinearities which impede further analytic investigation. However, the numerical solution of the above system unveils a sufficiently detailed picture of the resulting dynamics. For example, substituting back to equation (6), we obtain the function $\lambda(t)$. We now turn to the discussion of the results.

We start with the standard asymptotic analysis [10]. The system (7*a*)–(7*d*) exhibits one bifurcation at $\epsilon = 1$. This value separates two dynamical regimes with completely different asymptotic behaviour, illustrated in figure 1.

If $\epsilon > 1$ (*the ergodic regime*), both the mean position $\mu(t)$ and the mean velocity $\nu(t) = \langle \mathsf{V}(t)\rangle$ relax to zero. Asymptotically, the force switching exhibits the constant positive rate. We found $\lim_{t\to\infty}\lambda(t) = \gamma(\epsilon - 1)$, i.e. the mean force $\kappa(t)$ converges to zero (all the asymptotic values are approached exponentially). On the whole, the particle and the environment both thermalize. For the probability density for the particle coordinate, we obtain

$$\lim_{t\to\infty} p(x,t) = \frac{\gamma}{q}\frac{1}{B[(\epsilon-1)/2, 1/2]}(1-\tilde{x}^2)^{(\epsilon-3)/2}\Theta(1-\tilde{x}^2), \qquad (8)$$

where $\tilde{x} = x\gamma/q$, $\Theta(a)$ is the Heaviside unit-step function and $B(a, b)$ denotes the beta function [11]. For $\epsilon \in]1, 3[$ (for $\epsilon > 3$), the asymptotic density is a convex (concave) function. For $\epsilon = 3$, it is homogeneous within its support. In all, asymptotically, our model behaves as if the rate of switching is fixed from the very outset at the constant value $\frac{1}{2}\lambda_c = \gamma(\epsilon - 1)/2$.

If $\epsilon \in]0, 1[$ (*the non-ergodic regime*), the function $\lambda(t)$ converges to zero, i.e. the stochastic force gradually ceases to fluctuate. Asymptotically, the relaxation is exponential, the rate being γ for $\epsilon \in]0, \frac{1}{2}]$ and $2\gamma(1 - \epsilon)$ for $\epsilon \in [\frac{1}{2}, 1[$ (in the marginal case $\epsilon = 1$, $\lambda(t)$ approaches zero as $1/t$). As a result, the function $\Lambda(t)$ converges to an ϵ-dependent non-zero constant $\theta(\epsilon) = \int_0^\infty \mathrm{d}t\,\lambda(t)$, and the noise exhibits a *non-zero* mean value $\lim_{t\to\infty}\kappa(t) = q\delta_0 \exp[-\theta(\epsilon)]$. Notice that the time-asymptotic mean force depends on the initial conditions. If $x_0 = 0$, the value $\theta(\epsilon)$ is non-zero and characterizes the model. As for the particle, its mean position remains *frozen* out of the true equilibrium position. Actually, we get $\lim_{t\to\infty}\mu(t) = q\delta_0\xi(\epsilon)/\gamma$, where $\xi(\epsilon) = \exp[-\theta(\epsilon)]$. The asymptotic probability density for the particle position develops just two δ peaks at the points $x_\pm$, their weights being $\rho_\pm = [1 \pm \delta_0\xi(\epsilon)]/2$. In a sense, the constant $\xi(\epsilon)$ plays the role of an order parameter. We have numerically analysed its behaviour in the vicinity of the critical value $\epsilon = 1$. If $\epsilon \to 1^-$, the function $\xi(\epsilon)$ approaches zero as $(1-\epsilon)^{1/2}$.

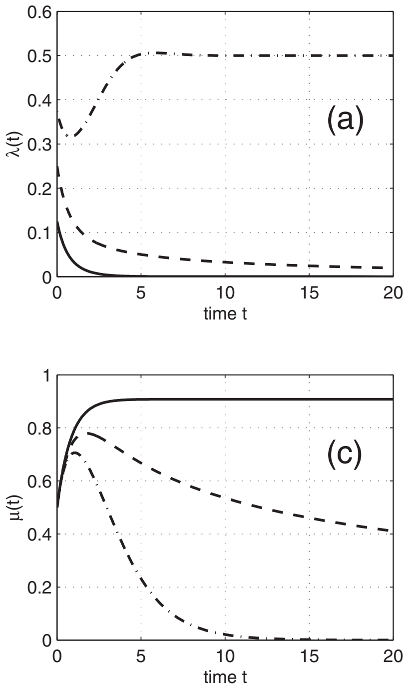

Figure 1. Time dependence of (a) the function $\lambda(t)$, (b) the mean force $\kappa(t) = \langle \mathsf{Q}(t)\rangle$, (c) the mean position $\mu(t) = \langle \mathsf{X}(t)\rangle$ and (d) the mean velocity $\nu(t) = \langle \mathsf{V}(t)\rangle$. In all panels, the full curves correspond to the control parameter $\epsilon = 0.5$ (the non-ergodic regime), the broken curves to $\epsilon = 1.0$, and the chain curves to $\epsilon = 1.5$ (the ergodic regime). Other parameters are $\gamma = 1.0$, and $q = 1.0$, in appropriate units. The initial conditions were $\delta_0 = 1.0$ and $x_0 = 0.5$.

Turning our attention to the stochastic properties of the resulting dynamics, we have analysed the position correlation function $c(t, t_w) = \langle \mathsf{X}(t)\mathsf{X}(t_w)\rangle$. Starting with its t derivative and with an appropriate initial condition at $t = t_w$, one arrives at a differential equation with the solution (we assume $x_0 = 0$ and $\tau = t - t_w \geqslant 0$)

$$c(t, t_w) = 2\exp(-\gamma\tau)s_4(t_w) + \frac{s_3(t_w)}{s_1(t_w)}[s_2(t) - \exp(-\gamma\tau)s_2(t_w)]. \tag{9}$$

Thus the discussion is again reduced to that of the system (7a)–(7d). In the ergodic regime, the position equilibrates. After a long enough waiting time t_w, the correlation function depends solely on the time difference $\tau = t - t_w$. We get

$$\lim_{t_w\to\infty} c(t_w + \tau, t_w) = \frac{q^2}{\gamma^2\epsilon(2-\epsilon)}\{\exp[-\gamma(\epsilon-1)\tau] - (\epsilon-1)\exp(-\gamma\tau])\}. \tag{10}$$

Contrary to this, in the non-ergodic regime, the limit yields the τ-independent value q^2/γ^2. After the waiting time t_w, the coordinate is already frozen and thus an additional increase of the time difference does not diminish the correlation. In other words, the Edwards–Anderson parameter $\lim_{\tau\to\infty}\lim_{t_w\to\infty} c(t, t_w)$ is equal to q^2/γ^2.

Finally, we have investigated the response of the particle to an external force. Adding the additional term $F_0\cos(\omega t)$ to the right-hand side of equation (1), one can repeat, *mutatis mutandis*, all steps leading to the system (7a)–(7d). The right-hand side of equation (6) then includes additional perturbation-dependent terms. Figure 2 presents a series of exact results based on the numerical solution of the system in question.

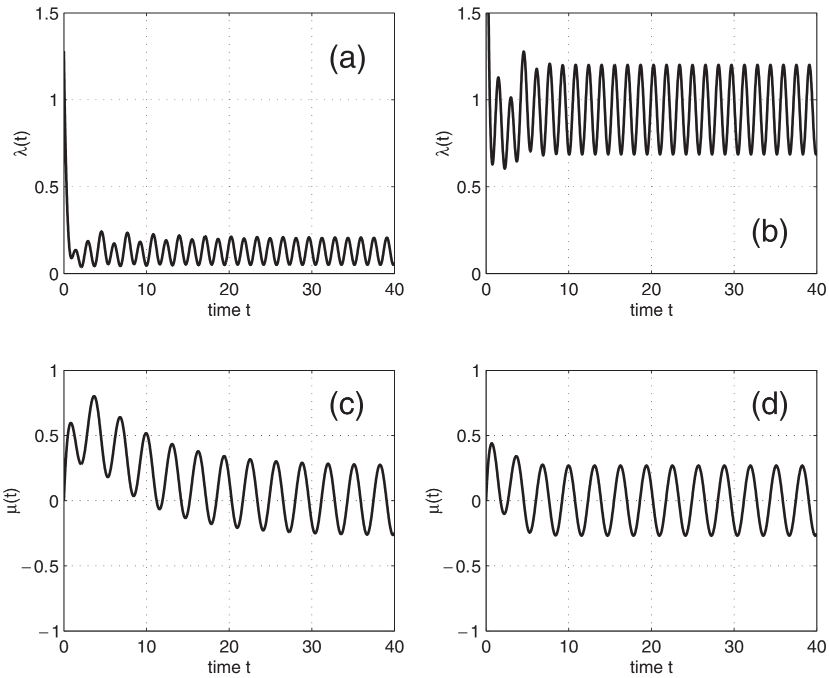

Figure 2. Exact response of the system to the external force $F_0 \cos(\omega t)$. The panels (a) and (c) present the function $\lambda(t)$ and the mean position $\mu(t)$ in the non-ergodic regime ($\epsilon = 0.5$). The panels (b) and (d) illustrate these functions in the ergodic regime ($\epsilon = 1.5$). Other parameters are $\gamma = 1.0$, $q = 1.0$, $F_0 = 0.6$ and $\omega = 2.0$, in appropriate units. The initial conditions were $\delta_0 = 1.0$ and $x_0 = 0.0$.

Focusing on the stationary regime, our calculation reveals two interesting points. First, for *any* $\epsilon > 0$, the mean position $\mu(t)$ oscillates around zero. The exact response is linear in the amplitude of the external force, the dynamical susceptibility being simply $\chi(t - t') = \Theta(t - t') \exp[-\gamma(t - t')]$.

Second, the function $\lambda(t)$ behaves as $\lambda(t) = A_0 + \sum_{k=1}^{\infty} A_k \sin(2k\omega t + \phi_k)$, i.e. it oscillates with the *doubled* frequency around a level A_0. The quantities A_0, A_k, ϕ_k, $k = 1, 2, \ldots$, are complicated functions of the external frequency ω and they are nonlinear functions of the amplitude F_0. Both observations can be easily understood. The external force drags the particle back and forth. Once moving, the particle induces through (3) the oscillations of the stochastic force. Once the force fluctuates, the two potential parabolas alternate and the particle can exploit the full set of accessible trajectories.

In conclusion, the coupling between particle motion and the stochastic force leads to a dynamical freezing in the non-ergodic regime. However, an arbitrarily small external periodic perturbation stimulates a 'melting' of the frozen state and the asymptotic response of the system looks like that in the ergodic regime. As a final remark, one can consider also other functional forms of the back-reaction mechanism. For example, our choice (3) can be generalized as $\lambda(t) = \langle F[\mathsf{V}^2(t)] \rangle$. We expect that our formalism would work provided $F(x)$ is an analytic function of its argument.

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